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Integrable Representations of Twisted Affine Lie Algebras

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INTRODUCTION

The purpose of this work is to study integrable modules for twisted affine Lie algebras. This extends the results concerning the non-twisted case obtained in our previous papers [2, 3].

The irreducible, integrable modules with finite-dimensional weight spaces for the non-twisted affine Lie algebras were classified in [2]. In the case when the centre acts non-trivially, it was found that the only such modules are the integrable highest weight modules and their duals. In fact, the same result holds in the twisted case, and the proof given in [2] requires no modification. We shall say no more about it here. In [3] explicit realizations were obtained for the modules on which the centre acts trivially. We recall the construction in order to state the results in the twisted case.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Fix a positive system for $(\mathfrak{g}, \mathfrak{h})$. Let $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p)$ be a p-tuple of dominant integral weights for $\mathfrak{h}$, and let $\mathbf{a} = (a_1, ..., a_p)$ be a p-tuple of non-zero complex numbers. If $V(\lambda)$ denotes the irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$, we make the loop space

$$L\left(\bigotimes_{j=1}^{p} V(\lambda_j)\right) = \bigotimes_{j=1}^{p} V(\lambda_j) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$$

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into a module for the loop algebra $L(\mathfrak{g})$ by defining

$$(f \cdot \Omega)(t) = \left(\sum_{j=1}^{p} 1 \otimes \cdots \otimes f(a_j t) \otimes \cdots \otimes 1 \right) \Omega(t)$$

for $f \in L(\mathfrak{g})$, $\Omega \in L(\otimes V(\lambda_j))$. Let d be the derivation

$$(d \cdot f)(t) = t \frac{df}{dt}$$

of $L(\mathfrak{g})$, and let $\bar{L}(\mathfrak{g}) = L(\mathfrak{g}) \tilde{\oplus} \mathbb{C} \cdot d$ be the corresponding semi-direct product Lie algebra. Choose $b \in \mathbb{C}$ and let d act on $L(\otimes V(\lambda_j))$ by

$$(d \cdot \Omega)(t) = b\Omega(t) + t \frac{d\Omega}{dt}.$$

It is easy to see that the resulting $\bar{L}(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is always integrable, but it is obviously reducible unless $a_1, ..., a_p$ are distinct. In that case, $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is usually, but not always, irreducible. In any case, it is completely reducible for $\bar{L}(\mathfrak{g})$, and it was shown in [3] that every irreducible, integrable $\bar{L}(\mathfrak{g})$-module occurs as a component of some $V(\boldsymbol{\lambda}, \mathbf{a}, b)$.

Let σ be a non-trivial diagram automorphism of $\mathfrak{g}$, k the order of σ, and ε a primitive kth root of unity. Define the twisted loop algebra

$$L_\sigma(\mathfrak{g}) = \{ f \in L(\mathfrak{g}) \colon \sigma(f(\varepsilon t)) = f(t) \}.$$

We show in Section 4 that each $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is completely reducible as an $\bar{L}_\sigma(\mathfrak{g})$-module. Moreover, every irreducible, integrable $\bar{L}_\sigma(\mathfrak{g})$-module occurs as a component of some $V(\boldsymbol{\lambda}, \mathbf{a}, b)$. In view of the results described above it is enough to show that every irreducible, integrable $\bar{L}_\sigma(\mathfrak{g})$-module is a subquotient of a $V(\boldsymbol{\lambda}, \mathbf{a}, b)$. This is done in Section 3.

To obtain explicit descriptions of the irreducible, integrable $\bar{L}_\sigma(\mathfrak{g})$-modules, we now only have to decompose the $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ under $\bar{L}_\sigma(\mathfrak{g})$. In fact, $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is irreducible for $\bar{L}_\sigma(\mathfrak{g})$ for a dense open set of values of $\mathbf{a} \in (\mathbb{C}^*)^p$. The decomposition in the general case is obtained in Section 4.

Remark. Together with the results of [2, 3], this completes the description of all irreducible, integrable modules with finite-dimensional weight spaces for affine Lie algebras. In [4], we construct the first examples of irreducible, integrable modules with infinite-dimensional weight spaces. These are obtained by taking tensor products of integrable highest weight modules and loop modules.

Until recently, the only known irreducible, integrable modules for non-affine Kac–Moody Lie algebras were the integrable highest weight modules and the adjoint representation. In [1], Borcherds has constructed some new irreducible, integrable modules for an arbitrary Kac–Moody Lie algebra. They all have finite-dimensional weight spaces.

Notation. We collect some standard notation to be used throughout the paper.

If $\mathfrak{m}$ is a finite-dimensional complex reductive Lie algebra, we denote by $\Delta(\mathfrak{m}, \mathfrak{a})$ the set of roots of $\mathfrak{m}$ with respect to a Cartan subalgebra $\mathfrak{a}$, and by $\pi(\mathfrak{m}, \mathfrak{a})$ a set of simple roots. Let $\Delta_+(\mathfrak{m}, \mathfrak{a})$ (resp. $\Delta_-(\mathfrak{m}, \mathfrak{a})$) be the corresponding set of positive (resp. negative) roots, and let $\check{\pi}(\mathfrak{m}, \mathfrak{a})$ be the set of simple coroots. Denote by μ_α the fundamental weight corresponding to $\alpha \in \Delta(\mathfrak{m}, \mathfrak{a})$. Let $\Delta^s(\mathfrak{m}, \mathfrak{a})$ (resp. $\Delta^l(\mathfrak{m}, \mathfrak{a})$) be the set of short (resp. long) roots. The meaning of symbols such as $\Delta^s_+(\mathfrak{m}, \mathfrak{a})$ is clear. If $\alpha \in \Delta(\mathfrak{m}, \mathfrak{a})$, then $\mathfrak{m}_\alpha$ denotes the root space of α. Finally, we write

$$\Gamma(\mathfrak{m}, \mathfrak{a}) = \left\{ \sum_{\alpha \in \pi(\mathfrak{m}, \mathfrak{a})} n_\alpha \alpha : n_\alpha \in \mathbb{Z} \right\}$$

for the root lattice of $\mathfrak{m}$ and set

$$\Gamma_+(\mathfrak{m}, \mathfrak{a}) = \left\{ \sum n_\alpha \alpha : n_\alpha \in \mathbb{Z}_+ \right\}$$

($\mathbb{Z}_+$ is the set of non-negative integers). If $\eta = \sum n_\alpha \alpha \in \Gamma_+(\mathfrak{m}, \mathfrak{a})$, set $\mathrm{ht}(\eta) = \sum n_\alpha$.

1. Twisted Loop Algebras

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of type A_n, D_n, or E_6, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let σ be a non-trivial diagram automorphism of $\mathfrak{g}$. Then σ has finite order k, where $k = 2$ when $\mathfrak{g}$ is of type A_n, D_n ($n \neq 4$), or E_6, and $k = 2$ or 3 when $\mathfrak{g}$ is of type D_4.

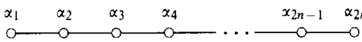
For many of the results in this paper, it is difficult to give a proof which covers the cases $k = 2$ and $k = 3$ simultaneously. We state all constructions and results in general, but give the proof only in the case $k = 2$. The proof in the remaining case is always a straightforward modification.

Let $\mathfrak{g}^0 \subset \mathfrak{g}$ be the subalgebra of fixed points of σ in $\mathfrak{g}$, and $\mathfrak{h}^0 = \mathfrak{h} \cap \mathfrak{g}^0$. It is proved in [6, Chap. X, Sect. 5; 7, Chap. 8] that $\mathfrak{g}^0$ is simple and $\mathfrak{h}^0$ is a Cartan subalgebra of $\mathfrak{g}^0$. Let $\Delta^0 = \Delta(\mathfrak{g}^0, \mathfrak{h}^0)$ and define π^0, $\check{\pi}^0$, $\Delta^0_\pm$, $(\Delta^0)^s$, $(\Delta^0)^l$ similarly.

LEMMA 1.1 [6, 7]. (a) *If* $\mathfrak{g}$ *is of type* A_{2n+1}, D_n, *or* E_6 *and* $k=2$, *then*

$$\pi^0 = \{(\alpha+\sigma(\alpha))|_{\mathfrak{h}^0} : \alpha \in \pi(\mathfrak{g}, \mathfrak{h}), \sigma(\alpha) \neq \alpha\}$$
$$\cup \{\alpha|_{\mathfrak{h}^0} : \alpha \in \pi(\mathfrak{g}, \mathfrak{h}), \sigma(\alpha) = \alpha\}.$$

(b) *If* $\mathfrak{g}$ *is of type* A_{2n}, *let* $\pi(\mathfrak{g}, \mathfrak{h}) = \{\alpha_1, ..., \alpha_{2n}\}$, *where the simple roots are numbered according to*



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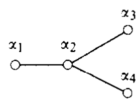
then $\pi^0 = \{(\alpha_i + \alpha_{2n-i+1})|_{\mathfrak{h}^0} : i \neq n\} \cup \{2(\alpha_n + \alpha_{n+1})|_{\mathfrak{h}^0}\}$.

(c) *If* $\mathfrak{g}$ *is of type* D_4 *and* $k=3$, *and the simple roots are numbered according to*



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then $\pi^0 = \{\alpha_1 + \alpha_3 + \alpha_4|_{\mathfrak{h}^0}\}$.

Similar statements hold for the set $\check{\pi}^0$ *of simple coroots.*

Define subalgebras $\mathfrak{n}_\pm$, $\mathfrak{n}^0_\pm$ of $\mathfrak{g}$ and $\mathfrak{g}^0$ respectively by

$$\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^0_\pm = \bigoplus_{\alpha \in \Delta^0_\pm} \mathfrak{g}^0_\alpha.$$

Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{g}^0 = \mathfrak{n}^0_- \oplus \mathfrak{h}^0 \oplus \mathfrak{n}^0_+$. Clearly $\mathfrak{n}^0_\pm \subseteq \mathfrak{n}_\pm$. In fact, $\mathfrak{n}^0_\pm$ is the subalgebra of fixed points of σ in $\mathfrak{n}_\pm$ and the notation is unambiguous.

Let $\mathfrak{g}^1 \subseteq \mathfrak{g}$ be the (-1)-eigenspace of σ in $\mathfrak{g}$ if $k=2$, or the sum of the $e^{\pm 2\pi i/3}$-eigenspaces if $k=3$. Then $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ and $[\mathfrak{g}^0, \mathfrak{g}^1] \subseteq \mathfrak{g}^1$. Set $\mathfrak{h}^1 = \mathfrak{h} \cap \mathfrak{g}^1$, $\mathfrak{n}^1_\pm = \mathfrak{n}_\pm \cap \mathfrak{g}^1$. Then $\mathfrak{g}^1 = \mathfrak{n}^1_+ \oplus \mathfrak{h}^1 \oplus \mathfrak{n}^1_-$. If $\lambda \in (\mathfrak{h}^0)^*$ let $\mathfrak{g}^1_\lambda = \{x \in \mathfrak{g}^1 : [h, x] = \lambda(h)x$ for all $h \in \mathfrak{h}^0\}$ be the corresponding weight space.

The following result determines the weights of $\mathfrak{g}^1$ as a $\mathfrak{g}^0$-module.

LEMMA 1.2. (a) *If* $\mathfrak{g}$ *is not of type* A_{2n}, *then*

$$\mathfrak{n}^1_\pm = \bigoplus_{\alpha \in (\Delta^0_\pm)^s} \mathfrak{g}^1_\alpha.$$

(b) *If* $\mathfrak{g}$ *is of type* A_{2n}, *then*

$$\mathfrak{n}^1_\pm = \bigoplus_{\alpha \in \Delta^0_\pm} \mathfrak{g}^1_\alpha \oplus \bigoplus_{\alpha \in (\Delta^0_\pm)^s} \mathfrak{g}^1_{2\alpha}.$$

In both cases, the weight spaces which appear are one-dimensional.

Proof. We sketch the proof, as it does not seem to be in the literature.

It is shown in [7, Chap. 8] that when $\mathfrak{g}$ is not of type A_{2n}, the highest weight of $\mathfrak{g}^1$ as a $\mathfrak{g}^0$-module is the highest short root θ of $\mathfrak{g}^0$. All conjugates of θ by the Weyl group of $\mathfrak{g}^0$ are therefore weights of $\mathfrak{g}^1$, and a dimension count shows that there are no others.

When $\mathfrak{g}$ is of type A_{2n}, the highest weight of $\mathfrak{g}^1$ as a $\mathfrak{g}^0$-module is 2θ. Now 2θ dominates the highest root ϕ of $\mathfrak{g}^0$, and also θ itself, so all conjugates of θ, ϕ, and 2θ are weights. Again a dimension count shows that there are no others.

Let $L=\mathbb{C}[t, t^{-1}]$ denote the algebra of Laurent polynomials in an indeterminate t. If V is a complex vector space, define $L(V)=V\otimes_{\mathbb{C}} L$. If we think of t as a parameter in the set $\mathbb{C}^*$ of non-zero complex numbers, then $L(V)$ is identified with the space of algebraic maps $\mathbb{C}^*\to V$. The vector space $L(V)$ has a natural $\mathbb{Z}$-grading:

$$L(V)=\bigoplus_{n\in\mathbb{Z}} V\otimes t^n.$$

If $\mathfrak{m}$ is any complex Lie algebra, then the loop space $L(\mathfrak{m})$ is a $\mathbb{Z}$-graded Lie algebra under pointwise operations. Define a derivation d of $L(\mathfrak{m})$ by

$$(d\cdot f)(t)=t\frac{df}{dt} \qquad (f\in L(\mathfrak{m})),$$

and let $\tilde{L}(\mathfrak{m})$ be the corresponding semi-direct product of $L(\mathfrak{m})$ with $\mathbb{C}\cdot d$. Explicitly, $\tilde{L}(\mathfrak{m})=L(\mathfrak{m})\oplus\mathbb{C}\cdot d$ with the bracket given by

$$[\Omega+\lambda d, \Omega'+\lambda' d]=[\Omega, \Omega']+\lambda\, d\Omega'-\lambda'\, d\Omega.$$

Fix a primitive kth root of unity ε, and extend the automorphism σ of $\mathfrak{g}$ to an automorphism of $\tilde{L}(\mathfrak{g})$ by

$$\sigma(d)=d, \qquad (\sigma(f))(t)=\sigma(f(\varepsilon t))$$

for $f\in L(\mathfrak{g})$. Let $L_\sigma(\mathfrak{g})$ (resp. $\tilde{L}_\sigma(\mathfrak{g})$) be the subalgebra of fixed points of σ on $L(\mathfrak{g})$ (resp. $\tilde{L}(\mathfrak{g})$). Then $L_\sigma(\mathfrak{g})$ is called the twisted loop algebra of $\mathfrak{g}$. For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ preserved by σ, define $L_\sigma(\mathfrak{a})$ and $\tilde{L}_\sigma(\mathfrak{a})$ similarly. Note that

$$\tilde{L}(\mathfrak{g})=L(\mathfrak{n}_-)\oplus\tilde{L}(\mathfrak{h})\oplus L(\mathfrak{n}_+)$$

and

$$\tilde{L}_\sigma(\mathfrak{g})=L_\sigma(\mathfrak{n}_-)\oplus\tilde{L}_\sigma(\mathfrak{h})\oplus L_\sigma(\mathfrak{n}_+).$$

We need a detailed description of the root systems of $\bar{L}(\mathfrak{g})$ and $\bar{L}_\sigma(\mathfrak{g})$. Introduce an element $\delta \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$ by

$$\delta|_{\mathfrak{h}} = 0, \qquad (\delta, d) = 1.$$

The restriction of δ to $\mathfrak{h}^0 \oplus \mathbb{C} \cdot d$ will also be denoted by δ. Identify $\mathfrak{h}^*$ (resp. $(\mathfrak{h}^0)^*$) with a subspace of $(\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$ by setting $(\lambda, d) = 0$ for $\lambda \in \mathfrak{h}^*$ (resp. $\lambda \in (\mathfrak{h}^0)^*$).

For the non-twisted loop algebra, set

$$L(\Delta) = \{\alpha + n\delta : \alpha \in \Delta, n \in \mathbb{Z}\}.$$

The sets $L(\Delta)_\pm$ are defined by replacing Δ by $\Delta_\pm$. For the twisted loop algebra, set

$$L_\sigma(\Delta) = \{\alpha + n\delta : \alpha \in (\Delta^0)^s, n \in \mathbb{Z}\} \cup \{\alpha + kn\delta : \alpha \in (\Delta^0)^l, n \in \mathbb{Z}\}$$

if $\mathfrak{g}$ is not of type A_{2n}, and

$$L_\sigma(\Delta) = \{\alpha + n\delta : \alpha \in \Delta^0, n \in \mathbb{Z}\} \cup \{2\alpha + (2n-1)\delta : \alpha \in (\Delta^0)^s, n \in \mathbb{Z}\}$$

if $\mathfrak{g}$ is of type A_{2n}. The sets $L_\sigma(\Delta)_\pm$ are defined in the obvious way.

The following result can be deduced easily from Lemma 1.2.

LEMMA 1.3. (a) $L(\mathfrak{g}) = \bigoplus_{\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*} L(\mathfrak{g})_\lambda$, *where* $L(\mathfrak{g})_\lambda = \{x \in L(\mathfrak{g}): [h, x] = \lambda(h)x$ *for all* $h \in \mathfrak{h} \oplus \mathbb{C} \cdot d\}$, *and* $L(\mathfrak{g})_\lambda \neq 0$ *if and only if* $\lambda \in L(\Delta) \cup \{n\delta : n \in \mathbb{Z}\}$. *In fact,*

$$L(\mathfrak{n}_\pm) = \bigoplus_{\lambda \in L(\Delta)_\pm} L(\mathfrak{g})_\lambda, \qquad \bar{L}(\mathfrak{h}) = \mathbb{C} \cdot d \oplus \bigoplus_{n \in \mathbb{Z}} L(\mathfrak{g})_{n\delta}$$

and all the root spaces $L(\mathfrak{g})_\lambda$ *are one-dimensional.*

(b) $L_\sigma(\mathfrak{g}) = \bigoplus_{\lambda \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*} L_\sigma(\mathfrak{g})_\lambda$, *where* $L_\sigma(\mathfrak{g})_\lambda$ *is defined in the obvious way, and* $L_\sigma(\mathfrak{g})_\lambda \neq 0$ *if and only if* $\lambda \in L_\sigma(\Delta) \cup \{n\delta : n \in \mathbb{Z}\}$. *In fact,*

$$L_\sigma(\mathfrak{n}_\pm) = \bigoplus_{\lambda \in L_\sigma(\Delta)_\pm} L_\sigma(\mathfrak{g})_\lambda, \qquad \bar{L}_\sigma(\mathfrak{h}) = \mathbb{C} \cdot d \oplus \mathfrak{h} \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} L_\sigma(\mathfrak{g})_{n\delta}$$

and all the root spaces $L_\sigma(\mathfrak{g})_\lambda$ *are one-dimensional.*

We conclude this section with a technical result which will be used in Section 3.

Let $\{x, y, h\}$ be the standard basis for $sl_2(\mathbb{C})$:

$$[h, x] = 2x, \; [h, y] = -2y, \; [x, y] = h.$$

Let $x_n = x \otimes t^n \in L(sl_2(\mathbb{C}))$, and define y_n, h_n similarly. For any integer $r \geqslant 1$, set $L^r = \mathbb{C}[t^r, t^{-r}]$ and $L^r(sl_2(\mathbb{C})) = sl_2(\mathbb{C}) \otimes_{\mathbb{C}} L^r$. Define $L^r(\mathbb{C}h)$ similarly.

PROPOSITION 1.4. *For integers $q, r \geqslant 1$, there exist polynomials $P_0, P_1, ..., P_q$ in the universal enveloping algebra $U(L^r(\mathbb{C}h))$, depending on q and r, such that*

$$x_r^q y_0^{q+1} = \sum_{j=0}^{q} y_{jr} P_{q-j} + g,$$

where $g \in U(L^r(sl_2(\mathbb{C})))$ is of the form

$$g = \sum_{i \in \mathbb{Z}} g_i x_{ir}$$

for some $g_i \in U(L^r(sl_2(\mathbb{C})))$. Furthermore, P_0 is a non-zero constant, and $P_i \in U(L^r(\mathbb{C}h))_{ir\delta}$.

Proof. For any $r \geqslant 1$, $L^r(sl_2(\mathbb{C}))$ is naturally isomorphic to $L(sl_2(\mathbb{C}))$ by sending $t \to t^r$. We may therefore assume $r = 1$. In this case, the proposition is a consequence of Lemma (7.5) of [5]. There, Garland obtains a formula for $y_1^q x_0^{q+1}$ and our formula is deduced from his by applying the automorphism of $sl_2(\mathbb{C})$ which interchanges x and y.

2. THE CATEGORIES $\tilde{\mathcal{O}}$ AND $\tilde{\mathcal{O}}^\sigma$

We define a category $\tilde{\mathcal{O}}^\sigma$ of $\bar{L}_\sigma(\mathfrak{g})$-modules analogous to the category $\tilde{\mathcal{O}}$ of $\bar{L}(\mathfrak{g})$-modules defined in Section 3 of [2]. We begin by recalling the definition of $\tilde{\mathcal{O}}$.

DEFINITION. An $\bar{L}(\mathfrak{g})$-module M lies in $\tilde{\mathcal{O}}$ if and only if

(i) M is a weight module, i.e.,

$$M = \bigoplus_{\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*} M_\lambda,$$

where $M_\lambda = \{m \in M : h \cdot m = \lambda(h) m \text{ for all } h \in \mathfrak{h} \oplus \mathbb{C} \cdot d\}$;

(ii) the set $P(M) = \{\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^* : M_\lambda \neq 0\}$ is contained in a finite union of cones $\tilde{D}(\mu) = \{\mu - \eta + n\delta : \eta \in \Gamma_+(\mathfrak{g}, \mathfrak{h}),\ n \in \mathbb{Z}\}$ for $\mu \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$.

Let $\mathcal{H}$ be the set of $\mathbb{Z}$-graded homomorphisms Λ: $U(L(\mathfrak{h})) \to L$ such that $\operatorname{im} \Lambda = \mathbb{C}[t^r, t^{-r}]$ for some $r \geqslant 0$. (The universal enveloping algebra $U(L(\mathfrak{h}))$ inherits its $\mathbb{Z}$-grading from that of $L(\mathfrak{h})$.) Fix $\Lambda \in \mathcal{H}$ and choose

$\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$ such that $\lambda|_{\mathfrak{h}} = \Lambda|_{\mathfrak{h}}$. If im $\Lambda = L^r$, we define an $\bar{L}(\mathfrak{h} \oplus \mathfrak{n}_+)$-module structure on L^r by

$$x \cdot t^{rs} = \Lambda(x) t^{rs}, \qquad d \cdot t^{rs} = (\lambda + rs\delta, d) t^{rs}, \qquad L(\mathfrak{n}_+) \cdot t^{rs} = 0$$

for all $x \in L(\mathfrak{h})$, $s \in \mathbb{Z}$. Set

$$M(\lambda, \Lambda) = U(\bar{L}(\mathfrak{g})) \bigotimes_{U(\bar{L}(\mathfrak{h} \oplus \mathfrak{n}_+))} L^r.$$

It is proved in Section 3 of [2] that $M(\lambda, \Lambda)$ has a unique irreducible quotient $V(\lambda, \Lambda)$. Moreover, $M(\lambda, \Lambda)$ is a free $L(\mathfrak{n}_-)$-module with basis $\{t^{rs}\}_{s \in \mathbb{Z}}$.

We now give the corresponding constructions for twisted loop algebras. Thus, $\tilde{\mathcal{O}}^\sigma$ will denote the category of $\bar{L}_\sigma(\mathfrak{g})$-modules M satisfying conditions (i) and (ii) above in which $\mathfrak{g}$ (resp. $\mathfrak{h}$, $\tilde{D}$, $\Gamma_+(\mathfrak{g}, \mathfrak{h})$) should be replaced by $\mathfrak{g}^0$ (resp. $\mathfrak{h}^0$, $\tilde{D}_\sigma$, $\Gamma_+(\mathfrak{g}^0, \mathfrak{h}^0)$). Further, we write $\mathscr{H}^\sigma$ for the set of $\mathbb{Z}$-graded homomorphisms $\Lambda^\sigma \colon U(L_\sigma(\mathfrak{h})) \to L$ with image L^r for some $r \geqslant 0$. Given $\Lambda^\sigma \in \mathscr{H}^\sigma$ with image L^r, choose $\lambda^\sigma \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*$ such that $\lambda^\sigma|_{\mathfrak{h}^0} = \Lambda^\sigma|_{\mathfrak{h}^0}$ and define an $\bar{L}_\sigma(\mathfrak{h} \oplus \mathfrak{n}_+)$-module structure on L^r as before. Set

$$M(\lambda^\sigma, \Lambda^\sigma) = U(\bar{L}_\sigma(\mathfrak{g})) \bigotimes_{U(\bar{L}_\sigma(\mathfrak{h} \oplus \mathfrak{n}_+))} L^r.$$

Then $M(\lambda^\sigma, \Lambda^\sigma)$ is a free $L_\sigma(\mathfrak{n}_-)$-module with basis $\{t^{rs}\}_{s \in \mathbb{Z}}$.

THEOREM 2.1. *Let $\lambda^\sigma \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*$, $\Lambda^\sigma \in \mathscr{H}^\sigma$ be such that $\lambda^\sigma|_{\mathfrak{h}^0} = \Lambda^\sigma|_{\mathfrak{h}^0}$. Then*:

(a) $M(\lambda^\sigma, \Lambda^\sigma) \in \tilde{\mathcal{O}}^\sigma$.

(b) $P(M(\lambda^\sigma, \Lambda^\sigma)) \subseteq \tilde{D}_\sigma(\lambda^\sigma)$. *Moreover, $\lambda^\sigma + s\delta \in P(M(\lambda^\sigma, \Lambda^\sigma))$ if and only if $s \equiv 0(r)$, in which case the weight space $M(\lambda^\sigma, \Lambda^\sigma)_{\lambda^\sigma + s\delta}$ is one-dimensional.*

(c) *$M(\lambda^\sigma, \Lambda^\sigma)$ has a unique irreducible quotient $V(\lambda^\sigma, \Lambda^\sigma)$.*

(d) *Every irreducible module in $\mathcal{O}^\sigma$ is of the form $V(\lambda^\sigma, \Lambda^\sigma)$.*

(e) *Suppose $V \in \tilde{\mathcal{O}}^\sigma$ contains a vector v such that*

$$V = U(\bar{L}_\sigma(\mathfrak{g})) \cdot v, \qquad h \cdot v = \lambda^\sigma(h) v \qquad \text{for all} \quad h \in \mathfrak{h}^0,$$
$$L_\sigma(\mathfrak{n}_+) \cdot v = 0, \qquad (\ker \Lambda^\sigma) \cdot v = 0.$$

Then V is a quotient of $M(\lambda^\sigma, \Lambda^\sigma)$ and $V(\lambda^\sigma, \Lambda^\sigma)$ is a quotient of V.

(f) *$V(\lambda^\sigma, \Lambda^\sigma) \cong V(\tilde{\lambda}^\sigma, \tilde{\Lambda}^\sigma)$ if and only if* $\ker \Lambda^\sigma = \ker \tilde{\Lambda}^\sigma$ *and $\tilde{\lambda}^\sigma = \lambda^\sigma + nr\delta$ for some $n \in \mathbb{Z}$, where* im Λ^σ $(=$ im $\tilde{\Lambda}^\sigma) = L^r$.

The corresponding result for $\tilde{\mathcal{O}}$ was obtained in [2] and the proof given there works without modification in the twisted case.

Let $v_\Lambda = 1 \otimes 1$ be the canonical generator of $M(\lambda, \Lambda)$ and let $\bar{v}_\Lambda$ be its image in $V(\lambda, \Lambda)$. Let $\bar{M}^\sigma$ (resp. $\bar{V}^\sigma$) be the $\bar{L}_\sigma(\mathfrak{g})$-submodule of $M(\lambda, \Lambda)$ (resp. $V(\lambda, \Lambda)$) generated by v_Λ(resp. $\bar{v}_\Lambda$).

COROLLARY 2.2. (a) *$\bar{M}^\sigma$ is isomorphic to $M(\lambda^\sigma, \Lambda^\sigma)$, where $\lambda^\sigma = \lambda|_{\mathfrak{h}^0 \oplus \mathbb{C}\cdot d}$ and $\Lambda^\sigma = \Lambda|_{U(L_\sigma(\mathfrak{h}))}$.*

(b) *$V(\lambda^\sigma, \Lambda^\sigma)$ is a quotient of $\bar{V}^\sigma$.*

Proof. (a) It is clear that $\bar{M}^\sigma$ is a quotient of $M(\lambda^\sigma, \Lambda^\sigma)$. That the two modules are isomorphic now follows from the fact that they are both $L_\sigma(\mathfrak{n}_-)$-free.

(b) This follows from part (e) of the theorem.

We recall the explicit construction of a family of modules in $\tilde{\mathcal{O}}$ given in [3].

For any $\lambda \in \mathfrak{h}^*$ let $V(\lambda)$ be the irreducible highest weight module for $\mathfrak{g}$ with highest weight λ. Fix an integer $p > 0$ and a complex number b. Choose p-tuples $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p) \in (\mathfrak{h}^*)^p$ and $\mathbf{a} = (a_1, ..., a_p) \in (\mathbb{C}^*)^p$, where the a_i are distinct. Define an action of $\bar{L}(\mathfrak{g})$ on the loop space $L(\otimes_{j=1}^p V(\lambda_j))$ as follows,

$$(f \cdot \Omega)(t) = \left(\sum_{j=1}^{p} 1 \otimes \cdots \otimes f(a_j t) \otimes \cdots \otimes 1 \right) \cdot \Omega(t)$$

$$(d \cdot \Omega)(t) = b\Omega(t) + t\frac{d\Omega}{dt},$$

for $f \in L(\mathfrak{g})$, $\Omega \in L(\otimes V(\lambda_j))$. Let $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ denote the corresponding $\bar{L}(\mathfrak{g})$-module.

Define a homomorphism $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}\colon U(L(\mathfrak{h})) \to L$ by extending

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(h \otimes t^n) = \left(\sum_{j=1}^{p} \lambda_j(h)\, a_j^n \right) t^n.$$

Then $\chi_{(\boldsymbol{\lambda}, \mathbf{a})} \in \mathscr{H}$.

The following result is contained in [3, Theorem (4.9)] and will be used in Section 3.

THEOREM 2.3. (a) *If $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ has image L^r then*

$$V(\boldsymbol{\lambda}, \mathbf{a}, b) \cong \bigoplus_{i=1}^{r} V(\lambda + i\delta, \chi_{(\boldsymbol{\lambda}, \mathbf{a})}),$$

where $\lambda = \sum_{j=1}^p \lambda_j + b\delta$.

(b) *Let* $\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*$, $\Lambda \in \mathscr{H}$ *be such that* $\lambda|_{\mathfrak{h}} = \Lambda|_{\mathfrak{h}}$. *Assume that there exist distinct* $a_1, ..., a_p \in \mathbb{C}^*$ *and complex numbers* $B_{\alpha,j}$ *for* $1 \leqslant j \leqslant p$, $\alpha \in \pi(\mathfrak{g}, \mathfrak{h})$, *such that*

$$\Lambda(\check{\alpha} \otimes t^n) = \left(\sum_{j=1}^{p} B_{\alpha,j} a_j^n \right) t^n$$

for all $\alpha \in \pi(\mathfrak{g}, \mathfrak{h})$, $n \in \mathbb{Z}$. *If* $b = (\lambda, d)$ *and* $\lambda_j = \sum_{\alpha \in \pi(\mathfrak{g},\mathfrak{h})} B_{\alpha,j} \mu_\alpha$ *then* $V(\lambda, \Lambda)$ *is a submodule of* $V(\boldsymbol{\lambda}, \mathbf{a}, b)$.

3. Classification of Integrable $\bar{L}_\sigma(\mathfrak{g})$-Modules

We begin our study of the integrable modules for the twisted loop algebra $\bar{L}_\sigma(\mathfrak{g})$.

Recall that a module V for a finite-dimensional complex reductive Lie algebra $\mathfrak{m}$ with Cartan subalgebra $\mathfrak{a}$ is said to be *integrable* if for every $v \in V$, $\alpha \in \Delta(\mathfrak{m}, \mathfrak{a})$ there exists an integer $k = k(\alpha, v) > 0$ such that $x^k \cdot v = 0$ for all $x \in \mathfrak{m}_\alpha$. It is well known that any such module V is a direct sum of finite-dimensional irreducible modules. In particular, V is a *weight module*, i.e., V is the sum of its weight spaces relative to $\mathfrak{h}$. (The weight spaces are not assumed to be finite-dimensional.)

One has a similar notion of integrable modules for the loop algebras $\bar{L}(\mathfrak{g})$ and $\bar{L}_\sigma(\mathfrak{g})$.

Definition 3.1. An $\bar{L}(\mathfrak{g})$-module (resp. $\bar{L}_\sigma(\mathfrak{g})$-module) V is said to be integrable if

(i) V is a weight module, i.e.,

$$V = \bigoplus_{\lambda \in (\mathfrak{h} \oplus \mathbb{C} \cdot d)^*} V_\lambda \quad \left(\text{resp. } V = \bigoplus_{\lambda \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*} V_\lambda \right);$$

(ii) for every $v \in V$, $\alpha \in L(\Delta)$ (resp. $\alpha \in L_\sigma(\Delta)$) there exists an integer $k = k(\alpha, v) > 0$ such that $x^k \cdot v = 0$ for all $x \in L(\mathfrak{g})_\alpha$ (resp. $L_\sigma(\mathfrak{g})_\alpha$).

Integrable $\bar{L}(\mathfrak{g})$-modules were studied in [2, 3]. We need the following results from those papers.

Theorem 3.2. (a) $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ *is an integrable* $\bar{L}(\mathfrak{g})$-*module if and only if each* λ_j *is dominant integral*; *i.e.*, $(\lambda_j, \check{\alpha}) \in \mathbb{Z}_+$ *for* $j = 1, ..., p$, $\alpha \in \pi(\mathfrak{g}, \mathfrak{h})$.

(b) $M(\lambda, \Lambda)$ *has an integrable quotient if and only if* $\Lambda = \chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ *and* $\lambda = \sum_{j=1}^{p} \lambda_j + b\delta$, *for some* $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p) \in (\mathfrak{h}^*)^p$, $\mathbf{a} = (a_1, ..., a_p) \in (\mathbb{C}^*)^p$, $b \in \mathbb{C}$.

Part (a) is easy (see [3, Proposition (2.5)]). Part (b) has the same proof as [2, Theorem (4.2)].

Let $\mathscr{I}_{\text{fin}}$ (resp. $\mathscr{I}^{\sigma}_{\text{fin}}$) denote the category of integrable $\bar{L}(\mathfrak{g})$-modules (resp. $\bar{L}_{\sigma}(\mathfrak{g})$-modules) with finite-dimensional weight spaces. It is obvious that if $V \in \mathscr{I}_{\text{fin}}$, then any $\bar{L}_{\sigma}(\mathfrak{g})$-submodule of V is in $\mathscr{I}^{\sigma}_{\text{fin}}$. Our work in this section and the following one is directed towards proving a converse of this statement:

THEOREM 3.3. *Let V^{σ} be an irreducible module in $\mathscr{I}^{\sigma}_{\text{fin}}$. Then there exists a module $V \in \mathscr{I}_{\text{fin}}$ such that V^{σ} is an $\bar{L}_{\sigma}(\mathfrak{g})$-submodule of V.*

An outline of the proof is as follows. We first prove that if $V^{\sigma} \in \mathscr{I}^{\sigma}_{\text{fin}}$ is irreducible, then $V^{\sigma} \in \tilde{\mathcal{O}}^{\sigma}$. By Theorem 2.1(d), this implies that $V^{\sigma} \cong V(\lambda^{\sigma}, \Lambda^{\sigma})$ for some $\lambda^{\sigma} \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*$, $\Lambda^{\sigma} \in \mathscr{H}^{\sigma}$ with $\lambda^{\sigma}|_{\mathfrak{h}^0} = \Lambda^{\sigma}|_{\mathfrak{h}^0}$. We then prove that $V(\lambda^{\sigma}, \Lambda^{\sigma})$ is a subquotient of an integrable $\bar{L}(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a}, b)$. The proof is completed in Section 4, where we show that $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ is completely reducible as an $\bar{L}_{\sigma}(\mathfrak{g})$-module.

We begin with the following result concerning integrable modules for a finite-dimensional complex reductive Lie algebra $\mathfrak{m}$.

LEMMA 3.4. *Let $\mathfrak{m}$ be a finite-dimensional complex reductive Lie algebra with Cartan subalgebra $\mathfrak{a}$. Let V be an integrable $\mathfrak{m}$-module with finite-dimensional weight spaces. Then:*

(a) *The set*

$$\begin{aligned} P_+(V) &= \{\lambda \in P(V) \colon \lambda + \eta \notin P(V) \text{ for all non-zero } \eta \in \Gamma_+(\mathfrak{m}, \mathfrak{a})\} \\ &= \{\lambda \in P(V) \colon \lambda + \alpha \notin P(V) \text{ for all } \alpha \in \Delta_+(\mathfrak{m}, \mathfrak{a})\} \end{aligned}$$

is non-empty.

(b) *All weights $\lambda \in P(V)$ are integral, i.e., $(\lambda, \check{\alpha}) \in \mathbb{Z}$ for all roots $\alpha \in \Delta(\mathfrak{m}, \mathfrak{a})$. Moreover, if $\lambda \in P_+(V)$ and $\alpha \in \Delta_+(\mathfrak{m}, \mathfrak{a})$ then $(\lambda, \check{\alpha}) \in \mathbb{Z}_+$ and $\mathfrak{m}_{\alpha} \cdot V_{\lambda} = 0$.*

(c) *Suppose $\lambda \in P(V)$, $\alpha \in \Delta(\mathfrak{m}, \mathfrak{a})$ are such that $(\lambda, \check{\alpha}) > 0$ (resp. $(\lambda, \check{\alpha}) < 0$). Then $\lambda - \alpha \in P(V)$ (resp. $\lambda + \alpha \in P(V)$).*

Part (a) was proved in [2, Lemma (2.6)]. Parts (b) and (c) are well known (see, for example, [7, Chap. 3]).

PROPOSITION 3.5. *If V^{σ} is an irreducible module in $\mathscr{I}^{\sigma}_{\text{fin}}$, then $V^{\sigma} \cong V(\lambda^{\sigma}, \Lambda^{\sigma})$ for some $\lambda^{\sigma} \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*$, $\Lambda^{\sigma} \in \mathscr{H}^{\sigma}$ such that $\lambda^{\sigma}|_{\mathfrak{h}^0} = \Lambda^{\sigma}|_{\mathfrak{h}^0}$.*

Proof. By Theorem 2.1(d), it suffices to prove that $V^{\sigma} \in \tilde{\mathcal{O}}^{\sigma}$. For this it is enough to find $\mu \in P(V^{\sigma})$ such that $L_{\sigma}(\mathfrak{n}_+) \cdot V^{\sigma}_{\lambda} = 0$, because the irreducibility of V^{σ}, together with the Poincaré–Birkhoff–Witt theorem, implies that $V^\sigma = U(\bar{L}_\sigma(\mathfrak{n}_- \oplus \mathfrak{h}^0)) \cdot V^\sigma_\mu$. But then $P(V^\sigma) \subseteq \tilde{D}_\sigma(\mu)$, and so $V^\sigma \in \tilde{\mathcal{C}}^\sigma$.

Now regard V^σ as a module for the finite-dimensional complex reductive Lie algebra $\mathfrak{m} = \mathfrak{g}^0 \oplus \mathbb{C} \cdot d$. By Lemma 3.4(a), the set $P_+(V^\sigma)$ is non-empty. Choose $\lambda \in P_+(V^\sigma)$. If $L_\sigma(\mathfrak{n}_+) \cdot V^\sigma_\lambda = 0$, then we can take $\mu = \lambda$ and we are through. Otherwise, there exists $\gamma \in L_\sigma(\Delta)_+$ with $\lambda + \gamma \in P(V^\sigma)$.

Claim 1. If $\nu \in P_+(V^\sigma)$ then $\nu + 2\alpha + (2n-1)\delta \notin P(V^\sigma)$ for any $\alpha \in (\Delta^0_+)^s$, $n \in \mathbb{Z}$.

Assuming Claim 1 for the present, γ must be of the form $\alpha + n\delta$ for some $\alpha \in \Delta^0_+$, $n \in \mathbb{Z}$. Choose γ such that if $\lambda + \beta + m\delta \in P(V^\sigma)$ for some $\beta \in \Delta^0_+$, $m \in \mathbb{Z}$ such that $\beta + m\delta \in L_\sigma(\Delta)_+$, then $\mathrm{ht}(\alpha) \geqslant \mathrm{ht}(\beta)$. We shall prove that $\mu = \lambda + \gamma$ has the required properties.

Claim 2. Let $\gamma' \in L_\sigma(\Delta)_+$ and assume that γ' is not of the form $2\alpha + (2n-1)\delta$ for $\alpha \in \Delta^0$, $n \in \mathbb{Z}$. Then $\lambda + \gamma + \gamma' \notin P(V^\sigma)$.

Assume the claim for the moment. If $\mathfrak{g}$ is not of type A_{2n}, the result follows immediately from Lemma 1.3(b). If $\mathfrak{g}$ is of type A_{2n}, we have to prove in addition that $\mu + \gamma'' \notin P(V^\sigma)$ for any $\gamma'' = 2\alpha + (2n-1)\delta$, $\alpha \in \Delta^0_+$, $n \in \mathbb{Z}$. Since every element of Δ^0_+ satisfies the hypotheses of Claim 2, it follows that $\mu \in P_+(V^\sigma)$. The result follows from Claim 1.

Proof of Claim 1. Put $\gamma' = \alpha + (2n-1)\delta$. Then

$$(\nu + 2\alpha + (2n-1)\delta, \check{\gamma}') = (\nu, \check{\alpha}) + 4 > 0,$$

so, by Lemma 3.4(c), $\nu + \alpha \in P(V^\sigma)$, contradicting $\nu \in P_+(V^\sigma)$.

Proof of Claim 2. We first make the following simple

Observation. Suppose that $\alpha + n\delta$, $\beta + m\delta \in L_\sigma(\Delta)_+$, where $\alpha, \beta \in \Delta^0_+$. Then

(a) either $\alpha + (m+n)\delta$ or $\beta + (m+n)\delta \in L_\sigma(\Delta)_+$;

(b) if $(\alpha, \check{\beta}) \leqslant 0$, then $\alpha + \beta + (m+n)\delta \in L_\sigma(\Delta)_+$.

Both parts are easily checked. (For (b) we use the fact that if $\alpha, \beta, \alpha + \beta \in \Delta^0_+$, then $\alpha + \beta$ is long if and only if both α and β are long.)

Suppose for a contradiction that $\lambda + \gamma + \gamma' \in P(V^\sigma)$ for some $\gamma' \in L_\sigma(\Delta)_+$ not of the form $2\alpha + (2n-1)\delta$. By Observation (a), either $\alpha + (m+n)\delta$ or $\beta + (m+n)\delta \in L_\sigma(\Delta)_+$. Suppose it is $\beta + (m+n)\delta = \gamma''$ (the proof in the other case is similar). By Observation (b), $(\alpha, \check{\beta}) \geqslant 0$ (otherwise we should have $\alpha + \beta + (m+n)\delta \in L_\sigma(\Delta)_+$, which contradicts the maximality of α). Hence $(\lambda + \alpha + \beta + (m+n)\delta, \gamma'') > 0$, and so $\lambda + \alpha \in P(V^\sigma)$, which contradicts $\lambda \in P_+(V^\alpha)$. This proves Claim 2.

The proof of the proposition is now complete.

The next two results identify every integrable module $V(\lambda^\sigma, \Lambda^\sigma)$ with a subquotient of some $V(\boldsymbol{\lambda}, \mathbf{a}, b)$. Let $\{\alpha_1, ..., \alpha_n\}$ be an enumeration of $\pi(\mathfrak{g}, \mathfrak{h})$ and define a permutation σ of $\{1, ..., n\}$ by $\sigma(\alpha_i) = \alpha_{\sigma(i)}$.

PROPOSITION 3.6. *Suppose that $\lambda^\sigma \in (\mathfrak{h}^0 \oplus \mathbb{C} \cdot d)^*$, $\Lambda^\sigma \in \mathscr{H}^\sigma$ are such that $V(\lambda^\alpha, \Lambda^\sigma)$ is integrable. Then we have the following.*

(a) *Assume $k=2$. There exist scalars $a_1, ..., a_m \in \mathbb{C}^*$ with $a_1^2, ..., a_m^2$ distinct, and for each i with $\sigma(i) \geqslant i$, scalars $B_{i,j} \in \mathbb{C}$, $j = 1, ..., 2m$, such that $B_{i,j} + B_{i,j+m} \in \mathbb{Z}_+$, and for all $n \in \mathbb{Z}$,*

$$\Lambda^\sigma(\check{\alpha}_i \otimes t^{2n}) = \left(\sum_{j=1}^{2m} B_{i,j} a_j^{2n}\right) t^{2n} \qquad \text{if} \quad \sigma(i) = i \tag{3.6i}$$

$$\Lambda^\alpha((\check{\alpha}_i + (-)^n \check{\alpha}_{\sigma(i)}) \otimes t^n) = \left(\sum_{j=1}^{2m} B_{i,j} a_j^n\right) t^n \qquad \text{if} \quad \sigma(i) > i, \tag{3.6ii}$$

where we put $a_{j+m} = -a_j$ for $j = 1, ..., m$.

(b) *Assume $\mathfrak{g}$ is of type D_4 and $k=3$. There exist scalars $a_1, ..., a_m \in \mathbb{C}^*$ with $a_1^3, ..., a_m^3$ distinct, and scalars $B_{1,j}, B_{2,j} \in \mathbb{C}$ for $j = 1, ..., 3m$ such that $B_{i,j} + B_{i,j+m} + B_{i,j+2m} \in \mathbb{Z}_+$ for $i = 1, 2$, and for all $n \in \mathbb{Z}$,*

$$\Lambda^\sigma(\check{\alpha}_2 \otimes t^{3n}) = \left(\sum_{j=1}^{3m} B_{2,j} a_j^{3n}\right) t^{3n}$$

$$\Lambda^\sigma(\varepsilon^{2n}\check{\alpha}_1 + \varepsilon^n \check{\alpha}_3 + \check{\alpha}_4) \otimes t^n) = \left(\sum_{j=1}^{3m} B_{1,j} a_j^n\right) t^n,$$

where we put $a_{j+m} = \varepsilon a_j$, $a_{j+2m} = \varepsilon^2 a_j$ for $j = 1, ..., m$. (The numbering of the simple roots is the same as in Lemma 1.1(c).)

Before proving the proposition we state the final result of this section. Let $\{\mu_1, ..., \mu_n\}$ be the set of fundamental weights of $\mathfrak{g}$ corresponding to the simple roots $\alpha_1, ..., \alpha_n$. Set

$$\lambda_j = \sum_i B_{i,j} \mu_i,$$

where the sum is over those indices i such that $\sigma(i) \geqslant i$ if $k=2$, and over $i = 1, 2$ if $k=3$. Define $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_{km})$, $\mathbf{a} = (a_1, ..., a_{km})$. Define also $\Lambda\colon U(L(\mathfrak{h})) \to L$ by

$$\Lambda(\check{\alpha}_i \otimes t^n) = \begin{cases} \left(\sum_{j=1}^{km} B_{i,j} a_j^n\right) t^n & \text{if } \sigma(i) \geqslant i \\ 0 & \text{otherwise} \end{cases}$$

(the sum is taken for $i=1,2$ if $k=3$). Set

$$\lambda = \sum_{j=1}^{km} \lambda_j.$$

By Theorem 2.3(b), $V(\lambda, \Lambda)$ is an $\bar{L}(\mathfrak{g})$-submodule of $V(\boldsymbol{\lambda}, \mathbf{a}, (\lambda, d))$. Also, since $\lambda|_{\mathfrak{h}^0} = \lambda^\sigma$, $\Lambda|_{U(L_\sigma(\mathfrak{h}))} = \Lambda^\sigma$, Corollary 2.2(b) implies that $V(\lambda^\sigma, \Lambda^\sigma)$ is a subquotient of $V(\lambda, \Lambda)$ and we have

COROLLARY 3.7. *With the definitions above, $V(\lambda^\sigma, \Lambda^\sigma)$ is a subquotient of $V(\boldsymbol{\lambda}, \mathbf{a}, (\lambda, d))$.*

Note that, in fact, $V(\lambda^\sigma, \Lambda^\sigma)$ is a quotient of the $\bar{L}_\sigma(\mathfrak{g})$-submodule of $V(\boldsymbol{\lambda}, \mathbf{a}, (\lambda, d))$ generated by $\tilde{\Omega}_{\boldsymbol{\lambda}}$, where $\tilde{\Omega}_{\boldsymbol{\lambda}} = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{km}}$, and v_{λ_j} is the highest weight vector in $V(\lambda_j)$.

We now show that, with the above choice of $\boldsymbol{\lambda}, \mathbf{a}$, the module $V(\boldsymbol{\lambda}, \mathbf{a}, (\lambda, d))$ is integrable for $\bar{L}(\mathfrak{g})$.

PROPOSITION 3.8. *If $V(\lambda^\sigma, \Lambda^\sigma)$ is integrable as an $\bar{L}_\sigma(\mathfrak{g})$-module, then $V(\boldsymbol{\lambda}, \mathbf{a}, (\lambda, d))$ is integrable as an $\bar{L}(\mathfrak{g})$-module.*

Combining this with Proposition 3.5 we obtain the main result of this section.

THEOREM 3.9. *Any irreducible module in $\mathscr{I}^\sigma_{\mathrm{fin}}$ is a subquotient of a module in $\mathscr{I}_{\mathrm{fin}}$.*

We now prove the last two propositions.

Proof of Proposition 3.6. We consider only part (a), where $k=2$.

Suppose first that $\sigma(i) = i$. If we regard $V(\lambda^\sigma, \Lambda^\sigma)$ as a module for the non-twisted loop algebra $L^2(\mathfrak{g}^0)$ then Theorem (4.1) of [2] implies the existence of scalars $a_{i,s} \in \mathbb{C}^*$, $C_{i,s} \in \mathbb{C}$, $s = 0, ..., s(i)$ say, with

$$\Lambda(\check{\alpha}_i \otimes t^n) = \left(\sum_{s=0}^{s(i)} C_{i,s} a_{i,s}^n \right) t^n$$

for all $n \in \mathbb{Z}$. (Note that the definition of integrability makes no reference to the action of d.)

If $\sigma(i) > i$, let $\{x_i, y_i, \check{\alpha}_i, i = 1, ..., n\}$ be a set of Chevalley generators for $\mathfrak{g}$. The subalgebra of $L_\sigma(\mathfrak{g})$ spanned by

$$\{(\check{\alpha}_i + \check{\alpha}_{\sigma(i)}) \otimes t^{2p}, (x_i + x_{\sigma(i)}) \otimes t^{2p}, (y_i + y_{\sigma(i)}) \otimes t^{2p}, i = 1, ..., n, p \in \mathbb{Z}\}$$

is isomorphic to $L(sl_2(\mathbb{C}))$. Let $(\lambda, \check{\alpha}_i + \check{\alpha}_{\sigma(i)}) = r(i) \in \mathbb{Z}_+$. Then by sl_2 theory,

$$(y_i + y_{\sigma(i)})^{r(i)+1} \cdot v_\Lambda = 0,$$

and hence

$$((x_i + (-)^n x_{\sigma(i)}) \otimes t^n)((x_i + x_{\sigma(i)}) \otimes t^2)^{r(i)} (y_i + y_{\sigma(i)})^{r(i)+1} \cdot v_\Lambda = 0$$

for all $n \in \mathbb{Z}$. By Proposition 1.4, there exist polynomials $P_{i,s} \in U(L^2(\mathbb{C}\check{\alpha}_i))$ for $s = 0, ..., r(i)$, such that

$$\sum_{s=0}^{r(i)} ((\check{\alpha}_i + (-)^n \check{\alpha}_{\sigma(i)}) \otimes t^{2s+n}) P_{i,r(i)-s} \cdot v_\Lambda = 0 \tag{1}$$

for all $n \in \mathbb{Z}$. Set

$$\Lambda((\check{\alpha}_i + (-)^n \check{\alpha}_{\sigma(i)}) \otimes t^n) = c_{i,n} t^n,$$
$$\Lambda(P_{i,s}) = A_{i,s} t^{2s}.$$

Then from (1) we obtain

$$\sum_{s=0}^{r(i)} c_{i,2s+n} A_{i,r(i)-s} = 0. \tag{2}$$

Note that $A_{i,0} \neq 0$, since $P_{i,0}$ is a non-zero constant.

Now we consider two cases. If $r(i) = 0$, then (2) implies that $c_{i,n} = 0$ for all $n \in \mathbb{Z}$, and we set $B_{i,s} = 0$ for all s.

If $r(i) > 0$, then $A_{i,s} \neq 0$ for some $s > 0$, for otherwise (2) implies $c_{i,n} = 0$ for all $n \in \mathbb{Z}$, and in particular $r(i) = (\lambda, \check{\alpha}_i + \check{\alpha}_{\sigma(i)}) = \Lambda(\check{\alpha}_i + \check{\alpha}_{\sigma(i)}) = c_{i,0} = 0$. Let $s(i)$ be the largest integer such that $A_{i,s(i)} \neq 0$, and let $a_{i,s} \in \mathbb{C}^*$, $s = 0, ..., s(i)$ be the set of roots of the polynomial

$$\sum_{s=0}^{s(i)} x^{2s} A_{i,s} + \sum_{s=0}^{s(i)} x^{2s+1} A_{i,s} = 0.$$

By Lemma B of Section 4 of [2], there exist scalars $C_{i,s} \in \mathbb{C}$ such that

$$c_{i,n} = \sum_{s=0}^{s(i)} C_{i,s} a_{i,s}^n$$

for all $n \in \mathbb{Z}$.

Consider the set of distinct elements in the set $\{a_{i,s} : \sigma(i) \geqslant i, s = 0, ..., s(i)\}$. By adjoining negatives if necessary, and reordering, we obtain a set of scalars $a_1, ..., a_m \in \mathbb{C}^*$ as in the statement of the proposition. Put $a_{j+m} = -a_j$ for $j = 1, ..., m$, and define

$$B_{i,j} = \sum_s C_{i,s}$$

for $j = 1, ..., 2m$, where the sum is over those indices s for which $a_{i,s} = a_j$ (and $B_{i,j} = 0$ if no $a_{i,s}$ is equal to a_j). This choice of the $B_{i,j}$ obviously satisfies (3.6i) and (3.6ii).

To prove the integrality conditions, consider the $L^2(\mathfrak{g}^0)$-module $\tilde{V} = U(L^2(\mathfrak{g}^0)) \cdot \bar{v}_A$. This is obviously integrable for $L^2(\mathfrak{g}^0)$, and

$$L^2(\mathfrak{n}^0_+) \cdot \bar{v}_A = 0, \qquad \ker(A\,|_{U(L^2(\mathfrak{h}^0))}) \cdot \bar{v}_A = 0.$$

By the non-twisted version of Theorem 2.1(e) (see [2]), $\tilde{V}$ is a quotient of the $L^2(\mathfrak{g}^0)$-module $M(\lambda\,|_{\mathfrak{h}^0}, A\,|_{U(L^2(\mathfrak{h}^0))})$. By Theorem (4.2) of [2], there exist integers $r_{i,p} \in \mathbb{Z}_+$ and distinct scalars $b_p \in \mathbb{C}^*$ such that

$$A(\check{\alpha}_i \otimes t^{2n}) = \sum_p r_{i,p} b_p^n \qquad \text{if} \quad \sigma(i) = i,$$

$$A((\check{\alpha}_i + \check{\alpha}_{\sigma(i)}) \otimes t^{2n}) = \sum_p r_{i,p} b_p^n \qquad \text{if} \quad \sigma(i) > i,$$

for all $n \in \mathbb{Z}$. Comparing with (3.6i) and (3.6ii), we see that the b_p must be equal to the a_j^2, $j = 1, ..., m$, in some order, and the corresponding $r_{i,p}$ must be equal to $B_{i,j} + B_{i,j+m}$.

Proof of Proposition 3.8. By Proposition (2.2) of [3], we must show that $B_{i,j} = (\lambda_j, \check{\alpha}_i) \in \mathbb{Z}_+$ for $\sigma(i) \geq i$, $j = 1, ..., m$. Assume, say, that $B_{i,1} \notin \mathbb{Z}_+$. We consider only the case $\sigma(i) > i$. The case $\sigma(i) = i$ is similar but easier.

Claim. For all $r \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, $y_i^r v_{\lambda_1} \oplus \cdots \oplus v_{\lambda_{2m}} \otimes t^n$ is an element of the $\bar{L}_\sigma(\mathfrak{g})$-module $W = \sum_{i \in \mathbb{Z}} U(\bar{L}_\sigma(\mathfrak{g})) \cdot \tilde{\Omega}_{\lambda,i}$.

Here $\tilde{\Omega}_{\lambda,i} = \tilde{\Omega}_\lambda \otimes t^i$.

Assume the Claim for the present. We compute

$$\begin{aligned}
&(x_i + x_{\sigma(i)})(y_i^r v_{\lambda_1} \otimes v_{\lambda_2} \otimes \cdots \otimes v_{\lambda_{2m}}) \\
&\quad = x_i(y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}) \\
&\quad = r((\lambda_1, \check{\alpha}_i) - r + 1)(y_i^{r-1} v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}) \\
&\quad = r(B_{i,1} - r + 1)(y_i^{r-1} v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}).
\end{aligned}$$

Iterating this calculation shows that

$$\begin{aligned}
&(x_i + x_{\sigma(i)})^r (y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}) \\
&\quad = r!\,(B_{i,1} - r + 1)(B_{i,1} - r + 2) \cdots B_{i,1} \tilde{\Omega}_\lambda.
\end{aligned}$$

As $B_{i,1} \notin \mathbb{Z}_+$, this is a non-zero multiple of $\tilde{\Omega}_\lambda$. Thus, for all $r \in \mathbb{Z}_+$, the $\bar{L}_\sigma(\mathfrak{g})$-module generated by $y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}$ contains $\tilde{\Omega}_\lambda$. This implies that $y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}}$ has non-zero image in the quotient $V(\lambda^\sigma, \Lambda^\sigma)$ of $U(\bar{L}_\sigma(\mathfrak{g})) \cdot \tilde{\Omega}_\lambda$. Hence, $\lambda - r\alpha_i \in P(V(\lambda^\sigma, \Lambda^\sigma))$ for all $r \in \mathbb{Z}_+$. Since $V(\lambda^\sigma, \Lambda^\sigma)$ is integrable as a $\mathfrak{g}^0$-module, its set of weights is invariant under the Weyl group of $\mathfrak{g}^0$. In particular, applying the simple reflection $s_{\alpha_i + \alpha_{\sigma(i)}}$, we find that $P(V(\lambda^\sigma, \Lambda^\sigma))$ contains

$$\begin{aligned} s_{\alpha_i + \alpha_{\sigma(i)}}(\lambda - r\alpha_i) = \lambda - r\alpha_i &- (\lambda, \check{\alpha}_i + \check{\alpha}_{\sigma(i)})(\alpha_i + \alpha_{\sigma(i)}) \\ &+ r(\alpha_i, \check{\alpha}_i + \check{\alpha}_{\sigma(i)})(\alpha_i + \alpha_{\sigma(i)}) \end{aligned} \tag{3}$$

for all $r \in \mathbb{Z}_+$. Inspecting Dynkin diagrams shows that $(\alpha_i, \check{\alpha}_i + \check{\alpha}_{\sigma(i)}) = 1$ or 2, so (3) contradicts the fact that $P(V(\lambda^\sigma, \Lambda^\sigma)) \subseteq \tilde{D}_\sigma(\lambda)$ (Theorem 2.2).

Proof of Claim (by induction on r). If $r = 0$ there is nothing to prove. Consider the equations

$$\begin{aligned} &((y_i + y_{\sigma(i)}) \otimes t^{2s})(y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}} \otimes t^{n-2s}) \\ &\quad = a_1^{2s}\{(y_i^{r+1} + y_{\sigma(i)} y_i^r) v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}} \\ &\qquad + y_i^r v_{\lambda_1} \otimes \cdots \otimes y_i v_{\lambda_{m+1}} \otimes \cdots \otimes v_{\lambda_{2m}}\} \otimes t^n \\ &\qquad + \sum_{j=2}^{m} a_j^{2s}(y_i^r v_{\lambda_1} \otimes \cdots \otimes y_i v_{\lambda_j} \otimes \cdots \otimes v_{\lambda_{2m}} \\ &\qquad + y_i^r v_{\lambda_1} \otimes \cdots \otimes y_i v_{\lambda_{j+m}} \otimes \cdots \otimes v_{\lambda_{2m}}) \otimes t^n \end{aligned}$$

for all $s \in \mathbb{Z}$. As $a_1^2, ..., a_m^2$ are distinct, this implies that

$$\begin{aligned} &((y_i^{r+1} + y_{\sigma(i)} y_i^r) v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}} \\ &\quad + y_i^r v_{\lambda_1} \otimes \cdots \otimes y_i v_{\lambda_{m+1}} \otimes \cdots \otimes v_{\lambda_{2m}}) \otimes t^n \end{aligned}$$

belongs to W. Similarly, by considering

$$((y_i - y_{\sigma(i)}) \otimes t^{2s+1})(y_i^r v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}} \otimes t^{n-2s-1})$$

we find that W contains

$$\begin{aligned} &((y_i^{r+1} - y_{\sigma(i)} y_i^r) v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_{2m}} \\ &\quad - y_i^r v_{\lambda_1} \otimes \cdots \otimes y_i v_{\lambda_{m+1}} \otimes \cdots \otimes v_{\lambda_{2m}} \otimes t^n. \end{aligned}$$

Adding these two elements of W completes the induction step.

4. Construction of the Integrable $\bar{L}_\sigma(\mathfrak{g})$-Modules

In this section we investigate the structure of $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ as an $\bar{L}_\sigma(\mathfrak{g})$-module. In particular, we prove that it is completely reducible, and describe explicitly its irreducible components. We assume that $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_p)$, where the λ_j are non-zero and dominant integral. We drop the b, since the action of d will be the same on all submodules of $V(\boldsymbol{\lambda}, \mathbf{a}, b) = V(\boldsymbol{\lambda}, \mathbf{a})$. We continue our policy of carrying out the proofs only in the case $k = 2$.

We begin with the following simple, but useful, result.

Proposition 4.1. (a) *As an $\bar{L}_\sigma(\mathfrak{g})$-module, $V(\boldsymbol{\lambda}, \mathbf{a})$ is isomorphic to a direct sum of modules of the form $V(\boldsymbol{\mu}, \mathbf{b})$, where the b_i^k are distinct.*

(b) *Suppose $\boldsymbol{\lambda}' \in (\mathfrak{h}^*)^p$, $\mathbf{a}' \in (\mathbb{C}^*)^p$ are such that for each i, either $\lambda_i' = \lambda_i$ and $a_i' = a_i$, or $\lambda_i = \sigma(\lambda_i)$ and $a_i' = \varepsilon^{-1} a_i$, or $\lambda_i' = \sigma^{-1}(\lambda_i)$ and $a_i' = \varepsilon a_i$. Then there is a canonical isomorphism of $\bar{L}_\sigma(\mathfrak{g})$-modules*

$$\mathscr{T}: V(\boldsymbol{\lambda}, \mathbf{a}) \to V(\boldsymbol{\lambda}', \mathbf{a}').$$

Proof. We begin with the following simple observation. For any $\lambda_1, \lambda_2 \in \mathfrak{h}^*$, define a new $(\mathfrak{g} \oplus \mathfrak{g})$-action on $V = V(\lambda_1) \otimes V(\lambda_2)$ by

$$(x, y) \cdot (v_1 \otimes v_2) = \begin{cases} xv_1 \otimes v_2 + v_1 \otimes yv_2 & \text{if} \quad x \in \mathfrak{g}^0 \\ -xv_1 \otimes v_2 + v_1 \otimes yv_2 & \text{if} \quad x \in \mathfrak{g}^1, \end{cases}$$

for $x, y \in \mathfrak{g}$, $v_i \in V(\lambda_i)$, $i = 1, 2$. One checks easily that this makes V into a $(\mathfrak{g} \oplus \mathfrak{g})$-module. It clearly has highest weight $(\sigma(\lambda_1), \lambda_2)$ and so contains a copy of $V(\sigma(\lambda_1)) \otimes V(\lambda_2)$. For dimensional reasons, V with the new action must be isomorphic to $V(\sigma(\lambda_1)) \otimes V(\lambda_2)$.

(a) Assume first that $p = 2$ and $\mathbf{a} = (-a, a)$. Make V into a $\mathfrak{g}$-module by restricting the $(\mathfrak{g} \oplus \mathfrak{g})$-action above to the diagonal, and write $x * v$ for the new action of $x \in \mathfrak{g}$ on $v \in V$. Then

$$(f\Omega)(t) = (f(-at) \otimes 1 + 1 \otimes f(at))\, \Omega(t) = f(at) * \Omega(t)$$

for all $f \in L_\sigma(\mathfrak{g})$, $\Omega \in L(V)$. Thus, if $V(\sigma(\lambda_1)) \otimes V(\lambda_2) \cong \oplus_\mu V(\mu)$ as $\mathfrak{g}$-modules, then $V(\boldsymbol{\lambda}, \mathbf{a}) \cong \oplus_\mu V(\mu, a)$ as $\bar{L}_\sigma(\mathfrak{g})$-modules.

The general case is dealt with in the same way. One groups together the pairs (a_i, a_j) with $a_i = -a_j$ and uses the above argument on each $V(\lambda_i) \otimes V(\lambda_j)$.

(b) It is enough to consider the case $p = 2$, $\mathbf{a}' = (-a_1, a_2)$, $\boldsymbol{\lambda}' = (\sigma(\lambda_1), \lambda_2)$. There is an isomorphism $\mathscr{T}$ between $V(\lambda_1) \otimes V(\lambda_2)$ with the new $(\mathfrak{g} \oplus \mathfrak{g})$-action defined above and $V(\sigma(\lambda_1)) \otimes V(\lambda_2)$, and by Schur's lemma, $\mathscr{T}$ is unique if we require that it preserve highest weight vectors. The corresponding map $\mathscr{T}: L(V(\lambda_1) \otimes V(\lambda_2)) \to L(V(\sigma(\lambda_1)) \otimes V(\lambda_2))$ is the desired isomorphism of $\bar{L}_\sigma(\mathfrak{g})$-modules $\mathscr{T}\colon V(\boldsymbol{\lambda}, \mathbf{a}) \to V(\boldsymbol{\lambda}', \mathbf{a}')$, where $\boldsymbol{\lambda}' = (\sigma(\lambda_1), \lambda_2)$, $\mathbf{a}' = (-a_1, a_2)$.

Part (a) means that we may and do *assume from now on that* $a_1^k, ..., a_p^k$ *are distinct.* There are now two cases to consider:

Case I. $\sigma(\lambda_j) \neq \lambda_j$ for some $j = 1, ..., p$.

Case II. $\sigma(\lambda_j) = \lambda_j$ for all $j = 1, ..., p$.

The behaviour of the $\bar{L}_\sigma(\mathfrak{g})$-module $V(\boldsymbol{\lambda}, \mathbf{a})$ is markedly different in the two cases.

If $\lambda \in \mathfrak{h}^*$ is any weight, we write $\lambda^0 = \lambda|_{\mathfrak{h}^0}$ (resp. $\lambda^1 = \lambda|_{\mathfrak{h}^1}$). Note that $\sigma(\lambda) = \lambda$ if and only if $\lambda^1 = 0$. Note also that if λ is non-zero and dominant, so is $\sigma(\lambda)$, and hence $\lambda^0 \neq 0$ (for $\lambda^0 = 0$ would imply that $\sigma(\lambda) = -\lambda$).

The first step is to analyze the character

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}\colon U(L_\sigma(\mathfrak{h})) \to L.$$

The argument used to prove Lemma (4.1) in [2] shows that its image is necessarily a Laurent subalgebra L^r of L for some $r \geqslant 1$.

PROPOSITION 4.2. *Assume that* $a_1^k, ..., a_p^k$ *are distinct. Then*:

(a) *In Case* I, $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}\colon U(L_\sigma(\mathfrak{h})) \to L$ *is surjective; i.e.,* $r = 1$.

(b) *In Case* II, $r = ks$ *for some* $s \geqslant 1$. *Moreover, the* λ_j *are then equal in groups of* s *and the corresponding* a_j^k *are proportional to the* sth *roots of unity.*

Proof (in the case $k = 2$). (a) Suppose that the image is L^r. Then r must be odd. For if r is even, then

$$\sum_{j=1}^{p} \lambda_j^1 a_j^{2n+1} = 0 \qquad \text{for all} \quad n \in \mathbb{Z}.$$

As the a_j^2 are distinct, this forces $a_j \lambda_j^1 = 0$ for all j, hence $\sigma(\lambda_j) = \lambda_j$ for all j, which contradicts the fact that we are in Case I.

Now suppose that $r > 1$. We claim that $p \equiv 0(r)$, and after reordering if necessary,

$$\lambda_1^0 = \cdots = \lambda_r^0, \lambda_{r+1}^0 = \cdots = \lambda_{2r}^0, ...,$$

$$a_1 \lambda_1^1 = \cdots = a_r \lambda_r^1, a_{r+1} \lambda_{r+1}^1 = \cdots = a_{2r} \lambda_{2r}^1, ...,$$

and

$$a_1^2, ..., a_r^2 = A_1 \rho, A_1 \rho^2, ..., A_1 \rho^r$$

$$a_{r+1}^2, ..., a_{2r}^2 = A_2 \rho, A_2 \rho^2, ..., A_2 \rho^r, ...,$$

for some $A_1, A_2, ..., \in \mathbb{C}^*$, where ρ is a primitive rth root of unity.

We know that

$$\sum_{j=1}^{p} \lambda_j^0 a_j^{2n} = 0 \qquad \text{if} \quad 2n \not\equiv 0(r),$$

i.e., if $n \not\equiv 0(r)$, and that all the λ_j^0 are non-zero. This implies the statements about the λ_j^0 and the a_j^2, by Lemma (4.5) of [3]. We also know that

$$\sum_{j=1}^{p} \lambda_j^1 a_j^{2n+1} = 0 \qquad \text{if} \quad 2n+1 \not\equiv 0(r).$$

Thus,

$$A_1^n(\rho^n a_1 \lambda_1^1 + \rho^{2n} a_2 \lambda_2^1 + \cdots + \rho^{rn} a_r \lambda_r^1) \\ + A_2^n(\rho^n a_{r+1} \lambda_{r+1}^1 + \cdots + \rho^{rn} a_{2r} \lambda_{2r}^1) + \cdots = 0,$$

if $2n+1 \not\equiv 0(r)$. Replacing n by $n+r$, $n+2r$, ..., and noting that A_1^r, A_2^r, ... are distinct (as the a_j^2 are distinct), we deduce that

$$\rho^n a_1 \lambda_1^1 + \rho^{2n} a_2 \lambda_2^1 + \cdots + \rho^{rn} a_r \lambda_r^1 = 0, \tag{4}$$

etc., if $2n+1 \not\equiv 0(r)$. Suppose $r = 2m+1$. Then $2n+1 \not\equiv 0(r)$ if $n = -m$, $-m+1$, ..., $m-1$. This is $2m = r-1$ distinct values of n, so (4) implies that

$$a_1 \lambda_1^1 = \cdots = a_r \lambda_r^1,$$

etc., and this proves the claim.

Finally, taking $n = 0$ in (4) gives $\sum_{j=1}^{r} a_j \lambda_j^1 = 0$, etc., and hence $\lambda_j^1 = 0$ for all j. This forces $\sigma(\lambda_j) = \lambda_j$ for all j, which is impossible, as we are in Case I. Thus, $r = 1$.

(b) In Case II, $\lambda_j^1 = 0$ for all j, and so it is obvious that the image is L^{2s} for some $s \geqslant 1$. We know that

$$\sum_{j=1}^{p} \lambda_j^0 a_j^{2n} = 0 \qquad \text{if} \quad n \not\equiv 0(s).$$

As usual, this implies that the λ_j^0 are equal in groups of s (and hence so are the λ_j themselves) and the corresponding a_j^2 are proportional to the sth roots of unity.

Note that it can happen that $s > 1$ in Case II, even when $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}: U(L(\mathfrak{h})) \to L$ is surjective. This occurs, for example, when $\boldsymbol{\lambda} = (\lambda, \lambda)$ with $\sigma(\lambda) = \lambda \neq 0$, and $\mathbf{a} = (1, i)$.

We now state the main result for Case I.

THEOREM 4.3. *Assume that we are in Case* I, *and that* $a_1^k, \ldots, a_p^k$ *are distinct. Then* $V(\boldsymbol{\lambda}, \mathbf{a})$ *is irreducible as an* $\bar{L}_\sigma(\mathfrak{g})$*-module.*

Proof. We prove first that, as an $\bar{L}_\sigma(\mathfrak{g})$-module, $V(\boldsymbol{\lambda}, \mathbf{a})$ is generated by $\tilde{\Omega}_{\boldsymbol{\lambda}}$. We assume that $p=2$ to keep the notation simple.

By Proposition 4.2(a), we can choose $Q, Q_* \in U(\bar{L}_\sigma(\mathfrak{h}))$ such that

$$\chi_{(\boldsymbol{\lambda},\mathbf{a})}(Q)=t, \qquad \chi_{(\boldsymbol{\lambda},\mathbf{a})}(Q_*)=t^{-1}.$$

Then

$$Q^n\tilde{\Omega}_{\boldsymbol{\lambda}}=\tilde{\Omega}_{\boldsymbol{\lambda},n}, \qquad Q_*^n\tilde{\Omega}_{\boldsymbol{\lambda}}=\tilde{\Omega}_{\boldsymbol{\lambda},-n}$$

for all $n \in \mathbb{Z}_+$. This proves that $\tilde{\Omega}_{\boldsymbol{\lambda},n} \in U(\bar{L}_\sigma(\mathfrak{g}))\cdot\tilde{\Omega}_{\boldsymbol{\lambda}}$ for all $n \in \mathbb{Z}$.

By an obvious induction argument, it suffices to prove that if $v_1 \otimes v_2 \otimes t^m \in U(\bar{L}_\sigma(\mathfrak{g}))\cdot\tilde{\Omega}_{\boldsymbol{\lambda}}$ for all $m \in \mathbb{Z}$, with $v_i \in V(\lambda_i)$, then the same is true for $yv_1 \otimes v_2 \otimes t^m$ for all $y \in \mathfrak{n}_-$. Write $y=y^0+y^1$ with $y^0 \in \mathfrak{n}_-^0$, $y^1 \in \mathfrak{n}_-^1$. Consider the equations

$$\begin{aligned}(y^0 \otimes t^{2s})&(v_1 \otimes v_2 \otimes t^{m-2s})\\ &=a_1^{2s}(y^0v_1 \otimes v_2 \otimes t^m)+a_2^{2s}(v_1 \otimes y^0v_2 \otimes t^m)\end{aligned}$$

for $s=0, 1$. Since $a_1^2 \neq a_2^2$, we find that $y^0v_1 \otimes v_2 \otimes t^m \in U(\bar{L}_\sigma(\mathfrak{g}))\cdot\tilde{\Omega}_{\boldsymbol{\lambda}}$. By considering $(y^1 \otimes t^{2s+1})(v_1 \otimes v_2 \otimes t^{m-2s-1})$, we find similarly that $y^1v_1 \otimes v_2 \otimes t^m \in U(\bar{L}_\sigma(\mathfrak{g}))\cdot\tilde{\Omega}_{\boldsymbol{\lambda}}$. Adding gives the result for y. This proves that $\tilde{\Omega}_{\boldsymbol{\lambda}}$ is an $\bar{L}_\sigma(\mathfrak{g})$-cyclic vector for $V(\boldsymbol{\lambda}, \mathbf{a})$.

To prove that $V(\boldsymbol{\lambda}, \mathbf{a})$ is irreducible, it is now enough to show that for every weight vector $\Omega \in V(\boldsymbol{\lambda}, \mathbf{a})$, there exists $g_\Omega \in U(\bar{L}_\sigma(\mathfrak{g}))$ such that $g_\Omega \cdot \Omega=\tilde{\Omega}_{\boldsymbol{\lambda},n}$ for some $n \in \mathbb{Z}$.

Observe that, by Theorem 2.1(e), $V(\boldsymbol{\lambda}, \mathbf{a})$ considered as an $\bar{L}_\sigma(\mathfrak{g})$-module is a quotient of $M(\lambda^0, \chi_{(\boldsymbol{\lambda},\mathbf{a})}|_{U(L_\sigma(\mathfrak{h}))})$. In particular, by Theorem 2.1(b), the weights of $V(\boldsymbol{\lambda}, \mathbf{a})$ as an $\bar{L}_\sigma(\mathfrak{g})$-module are of the form $\lambda^0-\eta+m\delta$, where $\eta \in \Gamma_+(\mathfrak{g}^0, \mathfrak{h}^0)$ and $m \in \mathbb{Z}$. We may assume that Ω has definite weight $\lambda^0-\eta+m\delta$.

We prove the existence of g_Ω as above by induction on $\operatorname{ht}(\eta)$. If $\operatorname{ht}(\eta)=0$ there is nothing to prove. Now assume that $\operatorname{ht}(\eta)>0$ and let

$$\Omega=\left(\sum_{r,s} c_{r,s}v_r \otimes w_s\right) \otimes t^m,$$

where $\{v_r\}$ and $\{w_s\}$ are bases of $V(\lambda_1)$ and $V(\lambda_2)$ consisting of $\mathfrak{h}$-weight vectors, and $c_{r,s} \in \mathbb{C}$. Using the induction hypothesis, it is enough to show that $(x \otimes t^{2p})\cdot\Omega \neq 0$ for some $x \in \mathfrak{n}_+^0$, $p \in \mathbb{Z}$, or $(x \otimes t^{2p+1})\cdot\Omega \neq 0$ for some

$x \in \mathfrak{n}_+^1$, $p \in \mathbb{Z}$. Assuming that $(x \otimes t^{2p})\Omega = 0$ for all $x \in \mathfrak{n}_+^0$, $p \in \mathbb{Z}$ implies that

$$a_1^{2p}\left(\sum c_{r,s} x v_r \otimes w_s\right) + a_2^{2p}\left(\sum c_{r,s} v_r \otimes x w_s\right) = 0$$

for all $p \in \mathbb{Z}$. As $a_1^2 \neq a_2^2$, this forces

$$\sum c_{r,s} x v_r \otimes w_s = 0, \qquad \sum c_{r,s} v_r \otimes x w_s = 0 \tag{5}$$

for all $x \in \mathfrak{n}_+^0$. Similarly, assuming that $(x \otimes t^{2p+1}) \cdot \Omega = 0$ for all $x \in \mathfrak{n}_+^1$, $p \in \mathbb{Z}$ implies that (5) must hold for all $x \in \mathfrak{n}_+^1$, and hence it must hold for all $x \in \mathfrak{n}_+$. Suppose that $c_{R,S} \neq 0$. The first equation in (5) implies that $x \cdot (\sum c_{r,S} v_r) = 0$ for all $x \in \mathfrak{n}_+$. Hence

$$\tilde{v} = \sum_r c_{r,S} v_r = a v_{\lambda_1}$$

for some $a \in \mathbb{C}^*$. Similarly,

$$\tilde{w} = \sum_s c_{R,s} w_s = b v_{\lambda_2}$$

for some $b \in \mathbb{C}^*$. As the v_r and w_s are weight vectors for $\mathfrak{h}$, this forces Ω to be a multiple of $\tilde{\Omega}_{\boldsymbol{\lambda},m}$, which contradicts $\operatorname{ht}(\eta) > 0$.

We now pass to Case II. By Propositions 4.1(b) and 4.2(b) we can assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_s,\ \lambda_{s+1} = \cdots = \lambda_{2s}, \ldots,$$
$$a_1, a_2, \ldots, a_s = A_1\rho, A_1\rho^2, \ldots, A_1\rho^s,$$
$$a_{s+1}, \ldots, a_{2s} = A_2\rho, A_2\rho^2, \ldots, A_2\rho^s, \ldots,$$

for some $A_1, A_2, \ldots \in \mathbb{C}^*$, where ρ is a primitive ksth root of unity. Put

$$\mathbf{a}' = (-A_1\rho, A_1\rho^2, \ldots, A_1\rho^{s-1}, A_1\rho^s, -A_2\rho, \ldots)$$

and let $\mathscr{T}$ be the isomorphism $V(\boldsymbol{\lambda}, \mathbf{a}) \to V(\boldsymbol{\lambda}, \mathbf{a}')$ constructed in the proof of Proposition 4.1(b). Let

$$\tau\colon \bigotimes_{j=1}^{p} V(\lambda_j) \to \bigotimes_{j=1}^{p} V(\lambda_j) = V$$

be the isomorphism which permutes the factors according to the permutation

$$(1\ 2 \cdots s)(s+1\ s+2 \cdots 2s) \cdots (p-s+1 \cdots p).$$

Define $T = \tau \cdot \mathscr{T}$. Then T^{ks} is the identity map on $V(\boldsymbol{\lambda}, \mathbf{a})$. Define

$$V^i(\boldsymbol{\lambda}, \mathbf{a}) = \{\Omega \in L(V) \colon \Omega(t) = \rho^i T\Omega(t)\}$$

(this notation should not be confused with that used in [3, Sect. 4]). Then clearly

$$V(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{i=1}^{ks} V^i(\boldsymbol{\lambda}, \mathbf{a}) \tag{$*$}$$

and it is easy to check that each $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is an $\bar{L}_\sigma(\mathfrak{g})$-submodule of $V(\boldsymbol{\lambda}, \mathbf{a})$. In fact, $(*)$ is the decomposition of $V(\boldsymbol{\lambda}, \mathbf{a})$ into its irreducible components.

THEOREM 4.4. *Assume that we are in Case* II, *and that* $a_1^k, ..., a_p^k$ *are distinct. Then each* $V^i(\boldsymbol{\lambda}, \mathbf{a})$ *is irreducible as an* $\bar{L}_\sigma(\mathfrak{g})$*-module.*

Proof (in the case $k = 2$). We prove first that $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is cyclically generated by $\tilde{\Omega}_{\boldsymbol{\lambda}, i}$. For this, it is enough to show that $V(\boldsymbol{\lambda}, \mathbf{a})$ is generated as an $\bar{L}_\sigma(\mathfrak{g})$-module by the vectors $\tilde{\Omega}_{\boldsymbol{\lambda}, i}$, $1 \leqslant i \leqslant 2s$.

Choose $Q, Q_* \in U(L_\sigma(\mathfrak{h}))$ such that

$$\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q) = t^{2s}, \qquad \chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q_*) = t^{-2s}.$$

Then

$$Q^n \tilde{\Omega}_{\boldsymbol{\lambda}, i} = \tilde{\Omega}_{\boldsymbol{\lambda}, i+2ns}, \qquad Q_*^n \tilde{\Omega}_{\boldsymbol{\lambda}, i} = \tilde{\Omega}_{\boldsymbol{\lambda}, i-2ns}$$

for all $n \in \mathbb{Z}_+$. This proves that the $\bar{L}_\sigma(\mathfrak{g})$-submodule generated by $\tilde{\Omega}_{\boldsymbol{\lambda}, i}$, $1 \leqslant i \leqslant 2s$, contains $\tilde{\Omega}_{\boldsymbol{\lambda}, n}$ for all $n \in \mathbb{Z}$. One can now repeat the proof of Theorem 4.3 to obtain the desired conclusion.

Finally, to prove that $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is irreducible, it is only necessary to observe that the proof of Theorem 4.3 can be repreated to prove that for every $\Omega \in V(\boldsymbol{\lambda}, \mathbf{a})$, there exists $g_\Omega \in U(\bar{L}_\sigma(\mathfrak{g}))$ such that $g_\Omega \cdot \Omega = \tilde{\Omega}_{\boldsymbol{\lambda}, n}$ for some $n \in \mathbb{Z}$. If $\Omega \in V^i(\boldsymbol{\lambda}, \mathbf{a})$, then $n \equiv i(2s)$ (since $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is a submodule) and by the first part of the proof it follows that $\Omega \in U(\bar{L}_\sigma(\mathfrak{g})) \cdot \tilde{\Omega}_{\boldsymbol{\lambda}, i}$.

Let us isolate the following consequence of Proposition 4.1(a) and Theorems 4.3 and 4.4.

COROLLARY 4.5. *Every integrable* $\bar{L}(\mathfrak{g})$*-module* $V(\boldsymbol{\lambda}, \mathbf{a}, b)$ *is completely reducible as an* $\bar{L}_\sigma(\mathfrak{g})$*-module.*

Together with Theorem 3.9, this completes the proof of Theorem 3.2. We have therefore constructed all the irreducible, integrable $\bar{L}_\sigma(\mathfrak{g})$-modules. To complete the picture, we have the following isomorphism classification theorem.

THEOREM 4.6. (a) *A module arising under Case* I *can never be isomorphic to one arising under Case* II.

(b) *In Case* I, $V(\boldsymbol{\lambda}, \mathbf{a}) \cong V(\boldsymbol{\mu}, \mathbf{b})$ *as* $\bar{L}_\sigma(\mathfrak{g})$*-modules if and only if there exist* $c \in \mathbb{C}^*$ *and a permutation* τ *of* $\{1, ..., p\}$ *such that for each* j *either* $\mu_{\tau(j)} = \lambda_j$ *and* $b_j = ca_j$, *or* $\mu_{\tau(j)} = \sigma(\lambda_j)$ *and* $b_j = \varepsilon^{-1} ca_j$, *or* $\mu_{\tau(j)} = \sigma^{-1}(\lambda_j)$ *and* $b_j = \varepsilon ca_j$.

(c) *In Case* II, $V^i(\boldsymbol{\lambda}, \mathbf{a}) \cong V^i(\boldsymbol{\mu}, \mathbf{a})$ *if and only if*

(i) $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ *and* $\chi_{(\boldsymbol{\mu}, \mathbf{b})}: U(L_\sigma(\mathfrak{h})) \to L$ *have the same image, say* L^{ks};

(ii) $i \equiv i'(ks)$;

(iii) *there exist* $c \in \mathbb{C}^*$ *and a permutation* τ *of* $\{1, 2, ..., p\}$ *such that, for each* j, $\mu_{\tau(j)} = \lambda_j$ *and* $b_j = ca_j$, εca_j *or* εca_j.

Proof (in the case $k = 2$).

(a) An isomorphism $V(\boldsymbol{\lambda}, \mathbf{a}) \to V^i(\boldsymbol{\mu}, \mathbf{b})$, where $\boldsymbol{\lambda}$ is in Case I and $\boldsymbol{\mu}$ is in Case II, must carry $\tilde{\Omega}_{\boldsymbol{\lambda}}$ (resp. $\tilde{\Omega}_{\boldsymbol{\lambda},1}$) onto a non-zero multiple of $\tilde{\Omega}_{\boldsymbol{\mu}}$ (resp. $\tilde{\Omega}_{\boldsymbol{\mu},1}$). But this is impossible since $\tilde{\Omega}_{\boldsymbol{\mu}}$ and $\tilde{\Omega}_{\boldsymbol{\mu},1}$ do not both belong to $V^i(\boldsymbol{\mu}, \mathbf{b})$.

(b) In one direction, this follows immediately from Proposition 4.1(b). Conversely, let i: $V(\boldsymbol{\lambda}, \mathbf{a}) \to V(\boldsymbol{\mu}, \mathbf{b})$ be an isomorphism of $L_\sigma(\mathfrak{g})$-modules. By Proposition 4.2(a), both $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}$ and $\chi_{(\boldsymbol{\mu}, \mathbf{b})}: U(L_\sigma(\mathfrak{h})) \to L$ are surjective. Choose $Q \in U(L_\sigma(\mathfrak{h}))$ such that $\chi_{(\boldsymbol{\lambda}, \mathbf{a})}(Q) = t$. Then $\chi_{(\boldsymbol{\mu}, \mathbf{b})}(Q) = ct$ for some $c \in \mathbb{C}$. We may assume that $i(\tilde{\Omega}_{\boldsymbol{\lambda}}) = \tilde{\Omega}_{\boldsymbol{\mu},1}$. Applying Q^n we find that $i(\tilde{\Omega}_{\boldsymbol{\lambda},n}) = c^n \tilde{\Omega}_{\boldsymbol{\mu},n}$ for all $n \in \mathbb{Z}_+$. This implies that $c \neq 0$ since i is injective. On the other hand, applying $h \otimes t^{2n}$ with $h \in \mathfrak{h}^0$ gives

$$c^{2n} \left(\sum \lambda_j(h)\, a_j^{2n} \right) \tilde{\Omega}_{\boldsymbol{\mu},2n} = \left(\sum \mu_j(h)\, b_j^{2n} \right) \tilde{\Omega}_{\boldsymbol{\mu},2n}$$

and hence

$$\sum \lambda_j^0 (ca_j)^{2n} = \sum \mu_j^0 b_j^{2n}$$

for all $n \in \mathbb{Z}_+$. This implies that up to a permutation,

$$\lambda_j^0 = \mu_j^0, \qquad b_j = \pm ca_j. \tag{6}$$

Next, applying $h \otimes t^{2n+1}$ with $h \in \mathfrak{h}^1$ gives

$$c^{2n+1} \left(\sum \lambda_j(h)\, a_j^{2n+1} \right) \tilde{\Omega}_{\boldsymbol{\mu},2n+1} = \left(\sum \mu_j(h)\, b_j^{2n+1} \right) \tilde{\Omega}_{\boldsymbol{\mu},2n+1}$$

and hence

$$\sum \lambda_j^1 (ca_j)^{2n+1} = \sum \mu_j^1 (b_j)^{2n+1}.$$

Thus,

$$\sum (\pm \lambda_j^1 b_j^{2n+1}) = \sum \mu_j^1 b_j^{2n+1}$$

with the same signs as in (6). As the b_j^2 are distinct, we must have $\pm \lambda_j^1 = \mu_j^1$ for all j. Since $\sigma(\lambda_j) = \lambda_j^0 - \lambda_j^1$ the proof is complete.

(c) This can be proved by an argument similar to that used in part (b), but it is easier to deduce it from the results in [3].

Suppose then that $V^i(\boldsymbol{\lambda}, \mathbf{a}) \cong V^{i'}(\boldsymbol{\mu}, \mathbf{b})$. It is clear by considering the action of d that $i \equiv i'(2s)$. Now regard both sides as modules for the non-twisted loop algebra $L^2(\mathfrak{g}^0)$. Theorem (4.9) of [3] implies that, up to a permutation, $\lambda_j^0 = \mu_j^0$, $a_j^2 = b_j^2$. This proves (iii).

The converse follows from Proposition 4.1(b), which implies that $V(\boldsymbol{\lambda}, \mathbf{a}) \cong V(\boldsymbol{\mu}, \mathbf{b})$. The first part of the proof shows that the isomorphism must carry $V^i(\boldsymbol{\lambda}, \mathbf{a})$ onto $V^i(\boldsymbol{\mu}, \mathbf{b})$.

We conclude this section by considering the question of unitarizability of the modules $V(\boldsymbol{\lambda}, \mathbf{a}, b)$.

Let θ be a compact conjugate-linear anti-involution of $\mathfrak{g}$, and extend θ to a conjugate-linear anti-involution of $\bar{L}(\mathfrak{g})$ by

$$\bar{\theta}(x \otimes t^n) = \theta(x) \otimes t^n \qquad (x \in \mathfrak{g}, n \in \mathbb{Z})$$

$$\theta(d) = d.$$

We write $\bar{\theta}(f) = f^*$ for $f \in \bar{L}(\mathfrak{g})$.

The first observation to make is that it is possible to choose θ so that it commutes with σ. In fact, in the notation of [6, pp. 505–507], we can take

$$\theta(X_i) = Y_i, \qquad \theta(Y_i) = X_i, \qquad \theta(H_i) = H_i.$$

In particular, θ then preserves the twisted subalgebra $\bar{L}_\sigma(\mathfrak{g}) \subset \bar{L}(\mathfrak{g})$, and σ preserves the compact form $\mathfrak{k}$ of $\mathfrak{g}$ defined by θ. We work with this choice of θ.

An $\bar{L}_\sigma(\mathfrak{g})$-module V is said to be *unitarizable* if there exists a positive definite Hermitian form $\langle\ ,\ \rangle$ on V such that

$$\langle f \cdot \Omega_1, \Omega_2 \rangle = \langle \Omega_1, f^* \cdot \Omega_2 \rangle$$

for all $\Omega_1, \Omega_2 \in V$, $f \in \bar{L}_\sigma(\mathfrak{g})$.

THEOREM 4.7. *Let $\lambda_1, \ldots, \lambda_p \in \mathfrak{h}^*$ be non-zero and dominant integral, and assume that $a_1^k, \ldots, a_p^k$ are distinct. Then the following are equivalent:*

(aI) *in Case* I, $V(\boldsymbol{\lambda}, \mathbf{a})$ *is unitarizable as an* $\bar{L}_\sigma(\mathfrak{g})$*-module*;

(aII) *in Case* II, $V^i(\boldsymbol{\lambda}, \mathbf{a})$ *is unitarizable as an* $\bar{L}_\sigma(\mathfrak{g})$*-module*;

(b) $a_1, ..., a_p$ *have the same modulus.*

Proof. That (b) implies (aI) and (aII) follows immediately from Theorem (3.2) of [3].

For (aI) $\Rightarrow$ (b), note that $V(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable as an $L^k(\mathfrak{g}^0)$-module. Theorem (3.2) of [3] implies that $a_1^k, ..., a_p^k$ have the same modulus, hence so do $a_1, ..., a_p$.

Finally, to prove (aII) $\Rightarrow$ (b), suppose, for example, that $V^0(\boldsymbol{\lambda}, \mathbf{a})$ is unitarizable. If $\langle\ ,\ \rangle_0$ is a positive definite Hermitian form on $V^0(\boldsymbol{\lambda}, \mathbf{a})$, let $c > 0$ be such that

$$c^{ks} = \frac{\|\tilde{\Omega}_{\lambda, ks}\|_0}{\|\tilde{\Omega}_\lambda\|_0}.$$

Noting that $V^i(\boldsymbol{\lambda}, \mathbf{a})$ is isomorphic to $V^0(\boldsymbol{\lambda}, \mathbf{a})$ as an $L_\sigma(\mathfrak{g})$-module under multiplication by z^i, we define a form $\langle\ ,\ \rangle_i$ on $V^i(\boldsymbol{\lambda}, \mathbf{a})$ by

$$\langle \Omega_1, \Omega_2 \rangle_i = c^i \langle z^{-1}\Omega_1, z^{-1}\Omega_2 \rangle_0$$

for $\Omega_1, \Omega_2 \in V^i(\boldsymbol{\lambda}, \mathbf{a})$. Then the direct sum form $\langle\ ,\ \rangle$ on $V(\boldsymbol{\lambda}, \mathbf{a}) = \bigoplus_{i=1}^{ks} V^i(\boldsymbol{\lambda}, \mathbf{a})$ is positive definite, Hermitian, and $L_\sigma(\mathfrak{g})$-invariant. The proof is completed as above. (Note that the proof of Theorem (3.2) in [3] assumed only that the form was $L(\mathfrak{g})$-invariant; the action of d was irrelevant.)

Let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, and $K \subset G$ the simply connected compact Lie group with Lie algebra $\mathfrak{k}$. Then σ lifts to automorphisms of K and G and the twisted loop groups $\bar{L}_\sigma(K)$ and $\bar{L}_\sigma(G)$ are defined in the obvious way as subgroups of $\bar{L}(K)$ and $\bar{L}(G)$.

THEOREM 4.8. (a) *Any irreducible, integrable* $\bar{L}_\sigma(\mathfrak{g})$*-module lifts to a representation of* $\bar{L}_\sigma(G)$.

(b) *Assume that* $\lambda_1, ..., \lambda_p$ *are non-zero and dominant integral, and that* $a_1^k, ..., a_p^k$ *are distinct. Then the following are equivalent*:

(i) *in Case I*, $V(\boldsymbol{\lambda}, \mathbf{a})$ *is a unitary representation of* $\bar{L}_\sigma(K)$;

(ii) *in Case* II, $V^i(\boldsymbol{\lambda}, \mathbf{a})$ *is a unitary representation of* $\bar{L}_\sigma(K)$;

(iii) $a_1, ..., a_p$ *have the same modulus.*

Proof. Part (a) is clear from the description of the irreducible, integrable $\bar{L}_\sigma(\mathfrak{g})$-modules obtained in this section. Part (b) follows from Theorem 4.7 in the same way in which, in [3], Theorem (3.5) followed from Theorem (3.2).

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